



Hardy-Sobolev Inequalities with Dunkl Weights

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Dedicated with great pleasure to Professor Duong Minh Duc, whose exemplary involvement with students deserves our greatest esteem, on the occasion of his 70th birthday

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Abstract

We establish several identities and inequalities of Hardy type and Sobolev type with Dunkl weights. We also investigate the sharp constants and optimal functions of the Hardy-Sobolev inequality with Dunkl weights.

Keywords Hardy inequality · Hardy-Sobolev inequality · Dunkl weight · Best constant

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1 Introduction

The Hardy inequalities and Sobolev inequalities are among the most used inequalities in analysis and play important roles in many areas of mathematical physics, partial differential equations and spectral theory. The main goal of this note is to study the optimal Hardy inequalities and Hardy-Sobolev inequalities with Dunkl weights.

The classical Hardy inequality states that for $N \geq 3$ and $u \in C_0^\infty(\mathbb{R}^N)$

$$\int_{\mathbb{R}^N} |\nabla u|^2 dx \geq \left| \frac{N-2}{2} \right|^2 \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^2} dx. \quad (1.1)$$

This is the multidimensional version of the following inequality that has been established in [18]: for nonnegative functions u

$$\int_0^\infty |u(x)|^2 dx \geq \frac{1}{4} \int_0^\infty \left(\frac{1}{x} \int_0^x u(t) dt \right)^2 dx.$$

It is well-known that the constant $\left| \frac{N-2}{2} \right|^2$ in (1.1) is optimal and cannot be attained by nontrivial functions. This fact has been well-studied in the literature. Actually, the problem of finding improved versions for Hardy's inequalities has attracted great attention. In particular, since the pioneering work of Brezis and Vázquez in [4], this question has been investigated extensively in a variety of contexts: by adding nonnegative terms to the RHS of (1.1), by replacing the usual ∇ by other operators, etc. The interested reader is referred to the monographs [2, 17, 19, 20, 26, 29, 31], to name just a few, that are standard references on the subject.

Weighted Hardy type inequalities have also been studied intensively in the literature. In particular, in [16], Frank and Seiringer proved the following general Hardy inequality by using the notion of the ground state representations: if the weighted p -Laplace equation, $p > 1$

$$-\operatorname{div} \left(V(x) |\nabla \varphi(x)|^{p-2} \nabla \varphi(x) \right) = W(x) \varphi(x)^{p-1}$$

has a positive weak solution φ , then

$$\int V(x) |\nabla u|^p dx \geq \int W(x) |u|^p dx.$$

To focus on the L^2 -Hardy type inequalities with radial weights, in [17], Ghoussoub and Moradifard made use of the notion of Bessel pairs and proved that if V and W are positive C^1 -functions on $(0, R)$ such that $(r^{N-1}V, r^{N-1}W)$ is a Bessel pair on $(0, R)$, $0 < R \leq \infty$, then

$$\int_{B_R} V(|x|) |\nabla u|^2 dx \geq \int_{B_R} W(|x|) |u|^2 dx \quad (1.2)$$

for all $u \in C_0^\infty(B_R \setminus \{0\})$. More interestingly, if (1.2) holds for all $u \in C_0^\infty(B_R \setminus \{0\})$, then $(r^{N-1}V, cr^{N-1}W)$ is a Bessel pair on $(0, R)$ for some $c > 0$. Here a couple of C^1 -functions (V, W) is a Bessel pair on $(0, R)$ if the ordinary differential equation

$$(Vy')' + Wy = 0$$

has a positive solution on the interval $(0, R)$. We refer the interested reader to the book [17] in which several examples and properties of Bessel pair have been studied. It is also worth pointing out that the Hardy type inequalities in the framework of equalities have also been investigated in [7, 12, 14, 23, 24], to mention but a few.

In [11], the authors introduced the notion of p -Bessel pairs and studied the best constants of the L^p -Hardy inequalities and the Hardy-Sobolev inequalities with monomial

weight $x^P = |x_1|^{P_1} \dots |x_N|^{P_N}$, $P = (P_1, \dots, P_N) \in (\mathbb{R}_{\geq 0})^N$. In particular, the following identities have been established in [11]:

Theorem Let $0 < R \leq \infty$, V and W be positive C^1 -functions on $(0, R)$. If $(r^{N+P_1+\dots+P_{N-1}}V, r^{N+P_1+\dots+P_{N-1}}W)$ is a (2-)Bessel pair on $(0, R)$, that is there exists $\varphi > 0$ on $(0, R)$ such that

$$\left(r^{N+P_1+\dots+P_{N-1}}V\varphi'\right)' + r^{N+P_1+\dots+P_{N-1}}W\varphi = 0.$$

Then, we have for all $u \in C_0^\infty(B_R^* \setminus \{0\})$ that

$$\begin{aligned} & \int_{B_R^*} V(|x|) |\nabla u|^2 x^P dx \\ &= \int_{B_R^*} W(|x|) |u|^2 x^P dx + \int_{B_R^*} V(|x|) \varphi^2(|x|) \left| \nabla \left(\frac{u}{\varphi(|x|)} \right) \right|^2 x^P dx \end{aligned}$$

and

$$\begin{aligned} & \int_{B_R^*} V(|x|) |\mathcal{R}u|^2 x^P dx \\ &= \int_{B_R^*} W(|x|) |u|^2 x^P dx + \int_{B_R^*} V(|x|) \varphi^2(|x|) \left| \mathcal{R} \left(\frac{u}{\varphi(|x|)} \right) \right|^2 x^P dx. \end{aligned}$$

Here $\mathcal{R} := \frac{x}{|x|} \cdot \nabla$ is the radial derivative, that is, in the polar coordinate $x = r\sigma$, $\mathcal{R} = \frac{\partial}{\partial r}$, and $B_R^* = B_R \cap \mathbb{R}_*^N$, $\mathbb{R}_*^N = \{(x_1, \dots, x_N) \in \mathbb{R}^N : x_i > 0 \text{ whenever } P_i > 0\}$.

The first primary goal of this article is to study the weighted Hardy inequalities with Dunkl weights, which are more general than the monomial weights. Therefore, we now will provide a brief introduction to Dunkl weight and Dunkl theory. For more details about Dunkl weight and Dunkl theory, we refer the interested reader to, for instance, [1, 30].

For $\alpha \in \mathbb{R}^N \setminus \{0\}$, we denote by σ_α the reflection with respect to the hyperplane $\langle \alpha \rangle^\perp$

$$\sigma_\alpha(x) = x - 2 \frac{\langle \alpha, x \rangle}{|\alpha|^2} \alpha, \quad x \in \mathbb{R}^N.$$

Let $R \subset \mathbb{R}^N \setminus \{0\}$ be a finite set such that $R \cap \mathbb{R}\alpha = \{\pm\alpha\}$ and $\sigma_\alpha(R) = R$ for all $\alpha \in R$. This is called a root system in Dunkl theory. The reflection group $G = G(R)$ is a finite group generated by all the reflections σ_α , $\alpha \in R$. Let $k : R \rightarrow \mathbb{R}_{\geq 0}$ be a multiplicity function, that is k is invariant under the nature action of G on R , i.e., $k(v) = k(u)$ whenever there exists $g \in G$ such that $gu = v$. We can write R as a disjoint union $R = R_+ \cup (-R_+)$ such that R_+ and $-R_+$ are separated by a hyperplane through the origin. Obviously, this decomposition is not unique. However, since k is G -invariant, the particular choice of R_+ does not make a difference in the below definition of Dunkl weight.

In this paper, we consider the Dunkl weight function $h_k(x)$ which is a G -invariant homogeneous weight of degree $2\gamma_k = \sum_{\alpha \in R_+} 2k(\alpha)$

$$h_k(x) = \prod_{\alpha \in R_+} |\langle \alpha, x \rangle|^{2k(\alpha)}.$$

This weight function $h_k(x)$ plays a key role in Dunkl theory and in the study of special functions with reflection symmetries. It is also worth noting that by choosing $R_+ = \{e_1, \dots, e_N\}$, then the Dunkl weight $h_k(x)$ becomes the monomial weight x^P .

Throughout the paper we assume that $k(\alpha) \geq 0$ and denote $d_k = N + 2\gamma_k$.

Our first result in this note is the following Hardy identities with Dunkl weight $h_k(x)$:

Theorem 1.1 *Let $N \geq 1$, $0 < R \leq \infty$, V and W be positive C^1 -functions on $(0, R)$. If $(r^{d_k-1}V, r^{d_k-1}W)$ is a Bessel pair on $(0, R)$, then for all $u \in C_0^\infty(B_R \setminus \{0\})$*

$$\begin{aligned} & \int_{B_R} V(|x|) |\nabla u|^2 h_k(x) dx - \int_{B_R} W(|x|) |u|^2 h_k(x) dx \\ &= \int_{B_R} V(|x|) \left| \nabla \left(\frac{u}{\varphi} \right) \right|^2 \varphi^2(|x|) h_k(x) dx \end{aligned}$$

and

$$\begin{aligned} & \int_{B_R} V(|x|) |\mathcal{R}u|^2 h_k(x) dx - \int_{B_R} W(|x|) |u|^2 h_k(x) dx \\ &= \int_{B_R} V(|x|) \left| \mathcal{R} \left(\frac{u}{\varphi} \right) \right|^2 \varphi^2(|x|) h_k(x) dx. \end{aligned}$$

Here $\varphi = \varphi_{N,k;V,W;R}$ is the positive solution of

$$\left(r^{d_k-1} V(r) y' \right)' + r^{d_k-1} W(r) y = 0$$

on $(0, R)$.

As a simple consequence of our Theorem 1.1, since $\left(r^{d_k-1}, r^{d_k-1} \left| \frac{d_k-2}{2} \right|^2 \frac{1}{|x|^2} \right)$ is a Bessel pair on $(0, \infty)$ with $\varphi(|x|) = |x|^{-\frac{d_k-2}{2}}$, we have the following Hardy inequality with Dunkl weight:

$$\begin{aligned} & \int_{\mathbb{R}^N} |\nabla u|^2 h_k(x) dx \\ &= \left| \frac{d_k-2}{2} \right|^2 \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^2} h_k(x) dx + \int_{\mathbb{R}^N} \left| \nabla \left(|x|^{\frac{d_k-2}{2}} u \right) \right|^2 \frac{h_k(x)}{|x|^{d_k-2}} dx \\ &\geq \left| \frac{d_k-2}{2} \right|^2 \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^2} h_k(x) dx. \end{aligned} \quad (1.3)$$

We will show later that this Hardy inequality with Dunkl weight can be used to derive the Sobolev inequality and Hardy-Sobolev inequality with Dunkl weight.

Our Theorem 1.1 can also be used to derive several Hardy type inequalities in the setting of equalities that generalized many known results in the literature. For instance, since $\left(r^{d_k-1} r^{2\varepsilon}, \left| \varepsilon + \frac{d_k-2}{2} \right|^2 r^{d_k-1} r^{2(\varepsilon-1)} \right)$ is a Bessel pair on $(0, \infty)$ with $\varphi(r) = r^{-\frac{d_k-2}{2}-\varepsilon}$, we obtain the following weighted Hardy type inequalities:

Corollary 1.1 *Let $\varepsilon \in \mathbb{R}$. We have for all $u \in C_0^\infty(\mathbb{R}^N \setminus \{0\})$ that*

$$\begin{aligned} \int_{\mathbb{R}^N} |x|^{2\varepsilon} |\nabla u|^2 h_k(x) dx &\geq \int_{\mathbb{R}^N} |x|^{2\varepsilon} |\mathcal{R}u|^2 h_k(x) dx \\ &\geq \left| \varepsilon + \frac{d_k-2}{2} \right|^2 \int_{\mathbb{R}^N} |x|^{2(\varepsilon-1)} |u|^2 h_k(x) dx. \end{aligned}$$

Another consequence of our Theorem 1.1 is that since $\left(r^{d_k-1}r^{2-d_k}, r^{d_k-1}r^{2-d_k}\frac{z_0^2}{R^2}\right)$ is a Bessel pair on $(0, R)$, $R > 0$, we obtain the following Poincaré identities and weighted Sobolev inequalities:

Corollary 1.2 *For $R > 0$ and $u \in C_0^\infty(B_R^* \setminus \{0\})$, there holds*

$$\begin{aligned} & \int_{B_R} \frac{|\nabla u|^2}{|x|^{d_k-2}} h_k(x) dx - \frac{z_0^2}{R^2} \int_{B_R} \frac{|u|^2}{|x|^{d_k-2}} h_k(x) dx \\ &= \int_{B_R} \frac{J_0^2\left(\frac{z_0}{R}|x|\right)}{|x|^{d_k-2}} \left| \nabla \left(\frac{u}{J_0\left(\frac{z_0}{R}|x|\right)} \right) \right|^2 h_k(x) dx, \\ & \int_{B_R} \frac{|\mathcal{R}u|^2}{|x|^{d_k-2}} h_k(x) dx - \frac{z_0^2}{R^2} \int_{B_R} \frac{|u|^2}{|x|^{d_k-2}} h_k(x) dx \\ &= \int_{B_R} \frac{J_0^2\left(\frac{z_0}{R}|x|\right)}{|x|^{d_k-2}} \left| \mathcal{R} \left(\frac{u}{J_0\left(\frac{z_0}{R}|x|\right)} \right) \right|^2 h_k(x) dx \end{aligned}$$

and

$$\int_{B_R} \frac{|\nabla u|^2}{|x|^{d_k-2}} h_k(x) dx \geq \int_{B_R} \frac{|\mathcal{R}u|^2}{|x|^{d_k-2}} h_k(x) dx \geq \frac{z_0^2}{R^2} \int_{B_R} \frac{|u|^2}{|x|^{d_k-2}} h_k(x) dx.$$

Moreover, the optimizer for the weighted Sobolev inequality

$$\frac{z_0^2}{R^2} \int_{B_R} \frac{|u|^2}{|x|^{d_k-2}} h_k(x) dx \leq \int_{B_R} \frac{|\nabla u|^2}{|x|^{d_k-2}} h_k(x) dx$$

exists and is of the form $J_0\left(\frac{z_0}{R}|x|\right)$.

More generally, if $R > 0$, $0 \leq \alpha \leq \frac{d_k-\lambda-2}{2}$, and z_α is the first zero of the Bessel function of the first kind $J_\alpha(z)$, then by using the Bessel pair $(r^{d_k-1}r^{-\lambda}, r^{d_k-1}r^{-\lambda} \left[\left(\frac{(d_k-\lambda-2)^2}{4} - \alpha^2 \right) \frac{1}{r^2} + \frac{z_\alpha^2}{R^2} \right])$ on $(0, R)$ with $\varphi(r) = r^{\frac{2-d_k+\lambda}{2}} J_\alpha\left(\frac{z_\alpha}{R}r\right)$, we get the following Brezis-Vázquez type identities and inequalities with Dunkl weight:

Corollary 1.3 *Let $R > 0$ and $0 \leq \alpha \leq \frac{d_k-\lambda-2}{2}$. For any $u \in C_0^\infty(B_R \setminus \{0\})$, there holds*

$$\begin{aligned} & \int_{B_R} \frac{|\nabla u|^2}{|x|^\lambda} h_k(x) dx \\ &= \left(\frac{(d_k-\lambda-2)^2}{4} - \alpha^2 \right) \int_{B_R} \frac{|u|^2}{|x|^{\lambda+2}} h_k(x) dx + \frac{z_\alpha^2}{R^2} \int_{B_R} \frac{|u|^2}{|x|^\lambda} h_k(x) dx \\ &+ \int_{B_R} \left| |x|^{\frac{2-d_k}{2}} J_\alpha\left(\frac{z_\alpha}{R}|x|\right) \nabla \left(\frac{u}{|x|^{\frac{2-d_k+\lambda}{2}} J_\alpha\left(\frac{z_\alpha}{R}|x|\right)} \right) \right|^2 h_k(x) dx, \end{aligned}$$

$$\begin{aligned}
& \int_{B_R} \frac{|\mathcal{R}u|^2}{|x|^\lambda} h_k(x) dx \\
&= \left(\frac{(d_k - \lambda - 2)^2}{4} - \alpha^2 \right) \int_{B_R} \frac{|u|^2}{|x|^{\lambda+2}} h_k(x) dx + \frac{z_\alpha^2}{R^2} \int_{B_R} \frac{|u|^2}{|x|^\lambda} h_k(x) dx \\
&+ \int_{B_R} \left| |x|^{\frac{2-d_k}{2}} J_\alpha \left(\frac{z_\alpha}{R} |x| \right) \mathcal{R} \left(\frac{u}{|x|^{\frac{2-d_k+\lambda}{2}} J_\alpha \left(\frac{z_\alpha}{R} |x| \right)} \right) \right|^2 h_k(x) dx,
\end{aligned}$$

and

$$\begin{aligned}
& \int_{B_R} \frac{|\nabla u|^2}{|x|^\lambda} h_k(x) dx \\
&\geq \int_{B_R} \frac{|\mathcal{R}u|^2}{|x|^\lambda} h_k(x) dx \\
&\geq \left(\frac{(d_k - \lambda - 2)^2}{4} - \alpha^2 \right) \int_{B_R} \frac{|u|^2}{|x|^{\lambda+2}} h_k(x) dx + \frac{z_\alpha^2}{R^2} \int_{B_R} \frac{|u|^2}{|x|^\lambda} h_k(x) dx.
\end{aligned}$$

Moreover, when $0 < \alpha \leq \frac{d_k - \lambda - 2}{2}$, the optimizer for the weighted Brezis-Vázquez inequality

$$\begin{aligned}
& \int_{B_R} \frac{|\nabla u|^2}{|x|^\lambda} h_k(x) dx \\
&\geq \left(\frac{(d_k - \lambda - 2)^2}{4} - \alpha^2 \right) \int_{B_R} \frac{|u|^2}{|x|^{\lambda+2}} h_k(x) dx + \frac{z_\alpha^2}{R^2} \int_{B_R} \frac{|u|^2}{|x|^\lambda} h_k(x) dx
\end{aligned}$$

exists and is of the form $|x|^{\frac{2-d_k+\lambda}{2}} J_\alpha \left(\frac{z_\alpha}{R} |x| \right)$.

For $\lambda < d_k - 2$, we have $\left(r^{d_k-1} \frac{1}{r^\lambda}, r^{d_k-1} \left(\frac{d_k-2-\lambda}{2} \right)^2 \frac{1}{r^{\lambda+2} (1 - (\frac{R}{r})^{-(d_k-2-\lambda)})^2} \right)$ is a Bessel pair on $(0, R)$ with $\varphi(r) = R^{-\frac{d_k-2-\lambda}{2}} \left(\left(\frac{r}{R} \right)^{-(d_k-2-\lambda)} - 1 \right)^{\frac{1}{2}} = \left(r^{-(d_k-2-\lambda)} - R^{-(d_k-2-\lambda)} \right)^{\frac{1}{2}}$. Hence, we obtain

Corollary 1.4 Let $\lambda < d_k - 2$ and $R > 0$. For any $u \in C_0^\infty(B_R \setminus \{0\})$, there holds

$$\begin{aligned}
& \int_{B_R} \frac{1}{|x|^\lambda} |\nabla u|^2 h_k(x) dx \\
&\geq \int_{B_R} \frac{1}{|x|^\lambda} |\mathcal{R}u|^2 h_k(x) dx \\
&\geq \left(\frac{d_k - 2 - \lambda}{2} \right)^2 \int_{B_R} \frac{1}{|x|^{\lambda+2} \left(1 - \left(\frac{R}{|x|} \right)^{-(d_k-2-\lambda)} \right)^2} |u|^2 h_k(x) dx.
\end{aligned}$$

In the limiting case $\lambda = d_k - 2$, we have that $(r^{d_k-1} \frac{1}{r^{d_k-2}}, r^{d_k-1} \frac{1}{4 r^{d_k} |\ln \frac{r}{R}|^2})$ is a Bessel pair on $(0, R)$ with $\varphi(r) = \left| \ln \frac{r}{R} \right|^{\frac{1}{2}}$. Therefore, as a consequence of our main results, we obtain the following critical Hardy inequalities with Dunkl weight:

Corollary 1.5 *For any $R > 0$ and $u \in C_0^\infty(B_R \setminus \{0\})$, there holds*

$$\begin{aligned} \int_{B_R} \frac{1}{|x|^{d_k-2}} |\nabla u|^2 h_k(x) dx &\geq \int_{B_R} \frac{1}{|x|^{d_k-2}} |\mathcal{R}u|^2 h_k(x) dx \\ &\geq \frac{1}{4} \int_{B_R} \frac{1}{|x|^{d_k} \left| \ln \frac{|x|}{R} \right|^2} |u|^2 h_k(x) dx. \end{aligned} \quad (1.4)$$

The constant $\frac{1}{4}$ in (1.4) is sharp, but is not attainable by nontrivial functions. See [10], for instance.

We note that we can regard Hardy's inequality as a linear inequality in the sense that it involves the square of u on both sides. In the same sense, the following Sobolev inequality and Hardy-Sobolev inequality can be viewed as a nonlinear inequality, where u appears with a higher power on the right than on the left side:

$$\begin{aligned} \left(\int_{\mathbb{R}^N} |\nabla u|^2 dx \right)^{\frac{1}{2}} &\geq S_{2,N} \left(\int_{\mathbb{R}^N} |u(x)|^{\frac{2N}{N-2}} dx \right)^{\frac{N-2}{2N}}, \\ \left(\int_{\mathbb{R}^N} |\nabla u|^2 dx \right)^{\frac{1}{2}} &\geq H S_{2,s,N} \left(\int_{\mathbb{R}^N} \frac{|u(x)|^{\frac{2(N-s)}{N-2}}}{|x|^s} dx \right)^{\frac{N-2}{2(N-s)}}. \end{aligned}$$

Therefore, we may think that the Sobolev inequality and Hardy-Sobolev inequality are stronger than the Hardy inequality. Remarkably, the authors in [11, 15, 17] showed that one can use Hardy inequality to derive the Sobolev inequality and Hardy-Sobolev inequality.

In the same line of thought, we will now show that non-optimal Sobolev inequality and Hardy-Sobolev inequality with Dunkl weight can be derived from our Hardy's inequality with Dunkl weight. More precisely, using the Hardy inequality with Dunkl weight $h_k(x)$ (1.3), we will prove the following Hardy-Sobolev type inequalities with Dunkl weight:

Theorem 1.2 *Let $0 \leq s < 2$. There exists a constant $C > 0$ such that for any $u \in C_c^1(\overline{\mathbb{R}_\epsilon^N})$, we have*

$$\left(\int_{\mathbb{R}_\epsilon^N} |\nabla u|^2 h_k(x) dx \right)^{\frac{1}{2}} \geq C \left(\int_{\mathbb{R}_\epsilon^N} |u(x)|^{\frac{2d_k}{d_k-2}} h_k(x) dx \right)^{\frac{d_k-2}{2d_k}} \quad (1.5)$$

and

$$\left(\int_{\mathbb{R}_\epsilon^N} |\nabla u|^2 h_k(x) dx \right)^{\frac{1}{2}} \geq C \left(\int_{\mathbb{R}_\epsilon^N} \frac{|u(x)|^{\frac{2(d_k-s)}{d_k-2}}}{|x|^s} h_k(x) dx \right)^{\frac{d_k-2}{2(d_k-s)}}. \quad (1.6)$$

Here $\mathbb{R}_\epsilon^N$ is a Weyl chamber. More precisely, let E be the set of all functions $\epsilon : R_+ \rightarrow \{-1, 1\}$, and for each $\epsilon \in E$ let

$$\mathbb{R}_\epsilon^N = \left\{ x \in \mathbb{R}^N : \operatorname{sgn}(\langle \alpha, x \rangle) = \epsilon(\alpha) \text{ for all } \alpha \in R_+ \right\}.$$

The connected components of $\mathbb{R}^N \setminus \{x \in \mathbb{R}^N : \langle \alpha, x \rangle = 0 \text{ for some } \alpha \in R\}$ are called the Weyl chambers associated to R . Weyl chambers are all of the form $\mathbb{R}_\epsilon^N$ for some $\epsilon \in E$. Moreover, the number of Weyl chambers equals the order of the reflection group $|G|$ since G acts simply transitively on the set of Weyl chambers.

We note that the Sobolev inequality (1.5) with Dunkl weight has been established via the Alexandroff-Bakelman-Pucci method in [5], via optimal transport in [21], using rearrangement in [33]. Best constant of (1.5) has also been computed explicitly in [33]. We also

mention that the weighted Sobolev and Hardy-Sobolev inequalities have been investigated extensively and intensively in the literature. See, for instance, [6, 8, 9, 22, 25, 27, 28, 34].

In many applications in analysis and geometry, one needs to know the precise information on the sharp constants and the optimizers of the functional and geometric inequalities. Therefore, our last goal of this article is to compute the sharp constant in the above Hardy-Sobolev inequality with Dunkl weights. More precisely, let $0 \leq s < 2$ and

$$HS_{2,s,k} = \inf \left\{ \frac{\int_{\mathbb{R}_\epsilon^N} |\nabla u|^2 h_k(x) dx}{\left(\int_{\mathbb{R}_\epsilon^N} \frac{|u(x)|^{2k(s)}}{|x|^s} h_k(x) dx \right)^{\frac{d_k-2}{d_k-s}}}; u \in C_c^1(\overline{\mathbb{R}_\epsilon^N}) \right\}.$$

We will study the optimizers and the best constant of (1.6):

Theorem 1.3 *We have*

$$HS_{s,N,k} = \left(\frac{1}{|G|} \frac{(2\pi)^{\frac{N}{2}}}{2^{\frac{d_k}{2}-1} \Gamma\left(\frac{d_k}{2}\right)} \prod_{\alpha \in R_+} \frac{\Gamma\left(k(\alpha) + \frac{1}{2} \langle \rho_k, \alpha \rangle + 1\right)}{\Gamma\left(\frac{1}{2} \langle \rho_k, \alpha \rangle + 1\right)} \right)^{\frac{2-s}{d_k-s}} \\ \times (d_k - 2)(d_k - s)^{\frac{d_k-2}{d_k-s}} \left[\frac{\Gamma\left(\frac{d_k-2}{2-s}\right) \Gamma\left(\frac{(d_k-s)p-d_k-2}{2-s}\right)}{\frac{2-s}{d_k-2} \Gamma\left(\frac{2(d_k-s)}{2-s}\right)} \right]^{\frac{2-s}{d_k-s}}.$$

Here $\rho_k = \sum_{\alpha \in R_+} k(\alpha) \alpha$. Also, $HS_{s,N,k}$ can be attained by functions of the form $u(x) = a(b + |x|^{2-s})^{-\frac{d_k-2}{2-s}}$, $a \in \mathbb{R}$, $b > 0$, $x \in \overline{\mathbb{R}_\epsilon^N}$.

2 Hardy Inequalities with Dunkl Weights—Proof of Theorem 1.1

Proof of Theorem 1.1 By polar coordinate, one has

$$\int_{B_R} B(|x|) |u|^2 h_k(x) dx = \int_{\mathbb{S}^{N-1}} h_k(\sigma) \int_0^R r^{d_k-1} B(r) |u|^2 dr d\sigma.$$

One notes that with $u = \varphi\psi$

$$\begin{aligned} \int_0^R r^{d_k-1} B(r) |u|^2 dr &= \int_0^R r^{d_k-1} B(r) |\varphi\psi|^2 dr \\ &= \int_0^R |\psi|^2 \varphi r^{d_k-1} B(r) \varphi dr \\ &= - \int_0^R |\psi|^2 \varphi \left(r^{d_k-1} A(r) \varphi' \right)' dr \\ &= \int_0^R r^{d_k-1} A(r) \varphi' \left(|\psi|^2 \varphi \right)' dr \\ &= \int_0^R r^{d_k-1} A(r) |\varphi'|^2 |\psi|^2 dr + 2 \int_0^R r^{d_k-1} A(r) \varphi' \varphi \psi \mathcal{R} \psi dr. \end{aligned}$$

Hence, by polar coordinate, one has

$$\begin{aligned}
 & \int_{B_R} B(|x|) |u|^2 h_k(x) dx \\
 &= \int_{\mathbb{S}^{N-1}} h_k(\sigma) \int_0^R r^{d_k-1} B(r) |u|^2 dr d\sigma \\
 &= \int_{\mathbb{S}^{N-1}} h_k(\sigma) \int_0^R r^{d_k-1} A(r) |\psi\varphi' + \varphi\partial_r\psi|^2 dr d\sigma \\
 &\quad - \int_{\mathbb{S}^{N-1}} h_k(\sigma) \int_0^R r^{d_k-1} A(r) |\varphi\partial_r\psi|^2 dr d\sigma \\
 &= \int_{\mathbb{S}^{N-1}} h_k(\sigma) \int_0^R r^{d_k-1} A(r) |\partial_r(\varphi\psi)|^2 dr d\sigma \\
 &\quad - \int_{\mathbb{S}^{N-1}} h_k(\sigma) \int_0^R r^{d_k-1} A(r) |\varphi\partial_r\psi|^2 dr d\sigma \\
 &= \int_{B_R} A(|x|) |\mathcal{R}u|^2 h_k(x) dx - \int_{B_R} A(|x|) \left| \mathcal{R}\left(\frac{u}{\varphi}\right) \right|^2 \varphi^2(|x|) h_k(x) dx.
 \end{aligned}$$

On the other hand,

$$\begin{aligned}
 & \int_{B_R} A(|x|) \left| \nabla\left(\frac{u}{\varphi}\right) \right|^2 \varphi^2(|x|) h_k(x) dx \\
 &= \int_{B_R} A(|x|) |\nabla u|^2 h_k(x) dx - \int_{B_R} A(|x|) |\psi\varphi'|^2 h_k(x) dx \\
 &\quad - 2 \int_{B_R} A(|x|) \psi\varphi'\varphi\mathcal{R}\psi h_k(x) dx.
 \end{aligned}$$

In the polar coordinate

$$\begin{aligned}
 & \int_{B_R} A(|x|) \psi\varphi'\varphi\mathcal{R}\psi h_k(x) dx \\
 &= \int_{\mathbb{S}^{N-1}} h_k(\sigma) \int_0^R A(r) \psi\varphi'\varphi\partial_r\psi r^{d_k-1} dr d\sigma.
 \end{aligned}$$

By integration by parts

$$\begin{aligned}
 - \int_0^R A(r) \psi\varphi'\varphi\partial_r\psi r^{d_k-1} dr &= \int_0^R \psi\partial_r \left(A(r) \psi\varphi' r^{d_k-1} \right) dr \\
 &= \int_0^R \psi \left[A(r) \partial_r\psi\varphi' r^{d_k-1} \right. \\
 &\quad \left. + A(r) \psi\varphi' r^{d_k-1} \varphi' \right. \\
 &\quad \left. + \psi\varphi \left(A(r) r^{d_k-1} \varphi' \right)' \right] dr.
 \end{aligned}$$

Hence,

$$\begin{aligned}
 & - \int_0^R A(r) |\psi\varphi'|^2 r^{d_k-1} dr - 2 \int_0^R A(r) \psi\varphi'\varphi\partial_r\psi r^{d_k-1} dr \\
 &= \int_0^R |\psi|^2 \varphi \left(A(r) r^{d_k-1} \varphi' \right)' dr \\
 &= - \int_0^R |\psi|^2 B(r) |\varphi|^2 r^{d_k-1} dr.
 \end{aligned}$$

In other words,

$$\begin{aligned} & - \int_{B_R} A(|x|) |\psi \varphi'|^2 h_k(x) dx - 2 \int_{B_R} A(|x|) \psi \varphi' \varphi \mathcal{R} \psi h_k(x) dx \\ & = - \int_{B_R} B(|x|) |u|^p h_k(x) dx. \end{aligned}$$

Therefore

$$\begin{aligned} & \int_{B_R} A(|x|) |\nabla u|^2 h_k(x) dx - \int_{B_R} B(|x|) |u|^2 h_k(x) dx \\ & = \int_{B_R} A(|x|) \left| \nabla \left(\frac{u}{\varphi} \right) \right|^2 \varphi^2(|x|) h_k(x) dx. \end{aligned} \quad \square$$

3 From Hardy Inequality with Dunkl Weight to Hardy-Sobolev Inequality with Dunkl Weight

We now will follow [33] to introduce the rearrangement argument with Dunkl weight. For any measurable subset $\Omega \subset \overline{\mathbb{R}_\epsilon^N}$, we define its rearrangement to be the set

$$\Omega^* = B_R(0) \cap \mathbb{R}_\epsilon^N.$$

Here $B_R(0)$ is the ball of radius R such that

$$\mu_k(\Omega^*) = \mu_k(\Omega)$$

with the measure

$$\mu_k(\Omega) = \int_{\Omega} h_k(x) dx.$$

For any measurable function $u : \overline{\mathbb{R}_\epsilon^N} \rightarrow \mathbb{R}$, we define its symmetric decreasing rearrangement $u^* : \overline{\mathbb{R}_\epsilon^N} \rightarrow \mathbb{R}_{\geq 0}$ by

$$u^*(x) = \int_0^\infty \chi_{\{|u|>t\}}^*(x) dt.$$

It can be checked that u^* is radial and decreasing, and it satisfies the property

$$\mu_k\left(\left\{x \in \mathbb{R}_\epsilon^N : |u| > t\right\}\right) = \mu_k\left(\left\{x \in \mathbb{R}_\epsilon^N : u^* > t\right\}\right), \quad \forall t > 0.$$

Therefore, by the layer cake representation, we have that if $\Psi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is nondecreasing, then $\int_{\mathbb{R}_\epsilon^N} \Psi(u) d\mu_k(x) = \int_{\mathbb{R}_\epsilon^N} \Psi(u^*) d\mu_k(x)$. Also, as in [11, 32, 33], since the isoperimetric inequality with Dunkl weight holds, for any Lipschitz continuous function u in $\mathbb{R}_\epsilon^N$ and any Young function Φ (that is, Φ maps $[0, \infty)$ into $[0, \infty)$, vanishes at 0, and is convex and increasing):

$$\int_{\mathbb{R}_\epsilon^N} \Phi(|\nabla u^*|) d\mu_k(x) \leq \int_{\mathbb{R}_\epsilon^N} \Phi(|\nabla u|) d\mu_k(x).$$

Proof of Theorem 1.2 Using the aforementioned rearrangement argument, to prove (1.5), we now can assume that u is a radial and decreasing function. Let us denote that

$$\mathbb{S}_\epsilon^{N-1} = \mathbb{R}_\epsilon^N \cap \mathbb{S}^{N-1}.$$

Then

$$\begin{aligned} \int_{\mathbb{R}_\epsilon^N} |u|^q h_k(x) dx &\geq \int_{B_R \cap \mathbb{R}_\epsilon^N} |u|^q h_k(x) dx = \int_{\mathbb{S}_\epsilon^{N-1}} h_k(\sigma) d\sigma \int_0^R |u|^q r^{2\gamma_k} r^{N-1} dr \\ &\geq |u(R)|^q \frac{R^{d_k}}{d_k} \int_{\mathbb{S}_\epsilon^{N-1}} h_k(\sigma) d\sigma. \end{aligned}$$

This implies

$$|u(x)| \leq d_k^{\frac{1}{q}} |x|^{-\frac{d_k}{q}} \left(\int_{\mathbb{R}_\epsilon^N} |u|^q h_k(x) dx \right)^{\frac{1}{q}} \left(\int_{\mathbb{S}_\epsilon^{N-1}} h_k(\sigma) d\sigma \right)^{-\frac{1}{q}}.$$

In particular, by choosing $q = \frac{2d_k}{d_k-2}$, we get

$$|u(x)| \leq d_k^{\frac{d_k-2}{2d_k}} \left(\int_{\mathbb{S}_\epsilon^{N-1}} h_k(\sigma) d\sigma \right)^{-\frac{d_k-2}{2d_k}} |x|^{-\frac{d_k-2}{2}} \left(\int_{\mathbb{R}_\epsilon^N} |u|^{\frac{2d_k}{d_k-2}} h_k(x) dx \right)^{\frac{d_k-2}{2d_k}}$$

and

$$|u(x)|^{\frac{4}{d_k-2}} \leq d_k^{\frac{2}{d_k}} \left(\int_{\mathbb{S}_\epsilon^{N-1}} h_k(\sigma) d\sigma \right)^{-\frac{2}{d_k}} |x|^{-2} \left(\int_{\mathbb{R}_\epsilon^N} |u|^{\frac{2d_k}{d_k-2}} h_k(x) dx \right)^{\frac{2}{d_k}}.$$

Equivalently,

$$\begin{aligned} &|u(x)|^{\frac{2d_k}{d_k-2}} h_k(x) \\ &\leq d_k^{\frac{2}{d_k}} \left(\int_{\mathbb{S}_\epsilon^{N-1}} h_k(\sigma) d\sigma \right)^{-\frac{2}{d_k}} |x|^{-2} \left(\int_{\mathbb{R}_\epsilon^N} |u|^{\frac{2d_k}{d_k-2}} h_k(x) dx \right)^{\frac{2}{d_k}} |u|^2 h_k(x). \end{aligned}$$

Therefore,

$$\begin{aligned} &\int_{\mathbb{R}_\epsilon^N} |u(x)|^{\frac{2d_k}{d_k-2}} h_k(x) dx \\ &\leq d_k^{\frac{2}{d_k}} \left(\int_{\mathbb{S}_\epsilon^{N-1}} h_k(\sigma) d\sigma \right)^{-\frac{2}{d_k}} \left(\int_{\mathbb{R}_\epsilon^N} |u|^{\frac{2d_k}{d_k-2}} h_k(x) dx \right)^{\frac{2}{d_k}} \int_{\mathbb{R}_\epsilon^N} \frac{|u|^2}{|x|^2} h_k(x) dx \\ &\leq \left| \frac{2}{d_k-2} \right|^2 d_k^{\frac{2}{d_k}} \left(\int_{\mathbb{S}_\epsilon^{N-1}} h_k(\sigma) d\sigma \right)^{-\frac{2}{d_k}} \left(\int_{\mathbb{R}_\epsilon^N} |u|^{\frac{2d_k}{d_k-2}} h_k(x) dx \right)^{\frac{2}{d_k}} \int_{\mathbb{R}_\epsilon^N} |\nabla u|^2 h_k(x) dx. \end{aligned}$$

In other words,

$$\begin{aligned} &\left(\int_{\mathbb{R}_\epsilon^N} |u(x)|^{\frac{2d_k}{d_k-2}} h_k(x) dx \right)^{\frac{d_k-2}{2d_k}} \\ &\leq \left| \frac{2}{d_k-2} \right| d_k^{\frac{1}{d_k}} \left(\int_{\mathbb{S}_\epsilon^{N-1}} h_k(\sigma) d\sigma \right)^{-\frac{1}{d_k}} \left(\int_{\mathbb{R}_\epsilon^N} |\nabla u|^2 h_k(x) dx \right)^{\frac{1}{2}}. \end{aligned}$$

As a consequence, we can also derive the Hardy-Sobolev inequality with Dunkl weight, which is the interpolation between the Hardy inequality and the Sobolev inequality:

$$HS_{s,N,k} \left(\int_{\mathbb{R}_\epsilon^N} \frac{|u(x)|^{2_k(s)}}{|x|^s} h_k(x) dx \right)^{\frac{2}{2_k(s)}} \leq \int_{\mathbb{R}_\epsilon^N} |\nabla u|^2 h_k(x) dx,$$

where $2_k(s) = \frac{2(d_k-s)}{d_k-2}$. Indeed, we have

$$\begin{aligned} & \int_{\mathbb{R}_\epsilon^N} \frac{|u(x)|^{2_k(s)}}{|x|^s} h_k(x) dx \\ &= \int_{\mathbb{R}_\epsilon^N} \frac{|u(x)|^s}{|x|^s} (h_k(x))^{\frac{s}{2}} |u(x)|^{2_k(s)-s} (h_k(x))^{\frac{2-s}{2}} dx \\ &\leq \left(\int_{\mathbb{R}_\epsilon^N} \left(\frac{|u(x)|^s}{|x|^s} (h_k(x))^{\frac{s}{2}} \right)^{\frac{2}{s}} dx \right)^{\frac{s}{2}} \left(\int_{\mathbb{R}_\epsilon^N} \left(|u(x)|^{2_k(s)-s} (h_k(x))^{\frac{2-s}{2}} \right)^{\frac{2}{2-s}} dx \right)^{\frac{2-s}{2}} \\ &= \left(\int_{\mathbb{R}_\epsilon^N} \frac{|u(x)|^2}{|x|^2} h_k(x) dx \right)^{\frac{s}{2}} \left(\int_{\mathbb{R}_\epsilon^N} |u(x)|^{\frac{2d_k}{d_k-2}} h_k(x) dx \right)^{\frac{2-s}{2}}. \end{aligned}$$

Using the Hardy inequality and Sobolev inequality with Dunkl weights, we get

$$\begin{aligned} & \left(\int_{\mathbb{R}_\epsilon^N} \frac{|u(x)|^2}{|x|^2} h_k(x) dx \right)^{\frac{s}{2}} \left(\int_{\mathbb{R}_\epsilon^N} |u(x)|^{\frac{2d_k}{d_k-2}} h_k(x) dx \right)^{\frac{2-s}{2}} \\ &\leq \left(\left| \frac{2}{d_k-2} \right|^2 \int_{\mathbb{R}_\epsilon^N} |\nabla u|^2 h_k(x) dx \right)^{\frac{s}{2}} \\ &\quad \times \left(\left| \frac{2}{d_k-2} \right|^{\frac{2d_k}{d_k-2}} d_k^{\frac{2}{d_k-2}} \left(\int_{\mathbb{S}_\epsilon^{N-1}} h_k(\sigma) d\sigma \right)^{-\frac{2}{d_k-2}} \left(\int_{\mathbb{R}_\epsilon^N} |\nabla u|^2 h_k(x) dx \right)^{\frac{1}{2} \frac{2d_k}{d_k-2}} \right)^{\frac{2-s}{2}} \\ &= \left| \frac{2}{d_k-2} \right|^s \left(\left| \frac{2}{d_k-2} \right|^{\frac{2d_k}{d_k-2}} d_k^{\frac{2}{d_k-2}} \left(\int_{\mathbb{S}_\epsilon^{N-1}} h_k(\sigma) d\sigma \right)^{-\frac{2}{d_k-2}} \right)^{\frac{2-s}{2}} \\ &\quad \times \left(\int_{\mathbb{R}_\epsilon^N} |\nabla u|^2 h_k(x) dx \right)^{\frac{s}{2}} \left(\left(\int_{\mathbb{R}_\epsilon^N} |\nabla u|^2 h_k(x) dx \right)^{\frac{d_k}{d_k-2}} \right)^{\frac{2-s}{2}} \\ &= \left| \frac{2}{d_k-2} \right|^s \left| \frac{2}{d_k-2} \right|^{\frac{d_k(2-s)}{d_k-2}} d_k^{\frac{2-s}{d_k-2}} \left(\int_{\mathbb{S}_\epsilon^{N-1}} h_k(\sigma) d\sigma \right)^{-\frac{2-s}{d_k-2}} \left(\int_{\mathbb{R}_\epsilon^N} |\nabla u|^2 h_k(x) dx \right)^{\frac{2_k(s)}{2}}. \end{aligned}$$

□

4 Optimal Constant of the Hardy-Sobolev Inequality—Proof of Theorem 1.3

First, we recall the Bliss lemma [3]:

Lemma 4.1 *Let m, n be constants, such that $n > m > 1$ and let $f(t)$ be a real single-valued nonnegative function in the interval $0 \leq t < \infty$ such that the integral*

$$J = \int_0^\infty f^m dt$$

taken in the sense of Lebesgue, is finite. Then the integral

$$F = \int_0^t f ds$$

is finite for every t and

$$I = \int_0^\infty \frac{F^n}{t^{n-r}} dt \leq \tau J^{\frac{n}{m}},$$

where $r = \frac{n}{m} - 1$, $\tau = \frac{1}{n-r-1} \left[\frac{r \Gamma(\frac{n}{r})}{\Gamma(\frac{1}{r}) \Gamma(\frac{n-1}{r})} \right]^r$. The equality happens for every function of the form $f = a(bt^r + 1)^{-\frac{r+1}{r}}$, and for every function differing from one of these only on a set of values of t of measure zero, but for no others.

Proof of Theorem 1.3 Let

$$HS_{2,s,k} = \inf \left\{ \frac{\int_{\mathbb{R}_\epsilon^N} |\nabla u|^2 h_k(x) dx}{\left(\int_{\mathbb{R}_\epsilon^N} \frac{|u(x)|^{2k(s)}}{|x|^s} h_k(x) dx \right)^{\frac{d_k-2}{d_k-s}}}; u \in C_c^1(\overline{\mathbb{R}_\epsilon^N}) \right\}.$$

By rearrangement argument and the Hardy-Littlewood inequality, we can assume that u is radial. In this case,

$$\int_{\mathbb{R}_\epsilon^N} |\nabla u|^2 h_k(x) dx = \left(\int_{\mathbb{S}_\epsilon^{N-1}} h_k(\sigma) d\sigma \right) \int_0^\infty |u_r|^2 r^{d_k-1} dr$$

and

$$\int_{\mathbb{R}_\epsilon^N} \frac{|u(x)|^{2k(s)}}{|x|^s} h_k(x) dx = \left(\int_{\mathbb{S}_\epsilon^{N-1}} h_k(\sigma) d\sigma \right) \int_0^\infty |u(r)|^{2k(s)} r^{d_k-1-s} dr.$$

Let $A = \int_0^\infty |u(r)|^{2_k(s)} r^{d_k-1-s} dr$ and $B = \int_0^\infty |u_r|^2 r^{d_k-1} dr$. Let $t = r^{2-d_k}$ (i.e., $r = t^{\frac{1}{2-d_k}}$) and $v(t) = u\left(t^{\frac{1}{2-d_k}}\right)$. Then

$$\begin{aligned} dt &= (2-d_k) r^{1-d_k} dr, \\ dr &= \frac{1}{2-d_k} t^{\frac{1}{2-d_k}-1} dt, \\ u_r &= v_t \frac{dt}{dr} \\ &= (2-d_k) v_t r^{1-d_k} \\ &= (2-d_k) v_t t^{\frac{1-d_k}{2-d_k}} \\ &= (2-d_k) v_t t^{1-\frac{1}{2-d_k}}. \end{aligned}$$

Therefore

$$\begin{aligned} A &= \int_0^\infty |u(r)|^{2_k(s)} r^{d_k-1-s} dr \\ &= - \int_0^\infty |v(t)|^{2_k(s)} t^{\frac{d_k-1-s}{2-d_k}} \frac{1}{2-d_k} t^{\frac{1}{2-d_k}-1} dt \\ &= \frac{1}{d_k-2} \int_0^\infty |v(t)|^{2_k(s)} t^{\frac{2d_k-s-2}{2-d_k}} dt \end{aligned}$$

and

$$\begin{aligned} B &= \int_0^\infty |u_r|^2 r^{d_k-1} dr \\ &= - \int_0^\infty \left| (2-d_k) v_t t^{1-\frac{1}{2-d_k}} \right|^2 t^{\frac{d_k-1}{2-d_k}} \frac{1}{2-d_k} t^{\frac{1}{2-d_k}-1} dt \\ &= (d_k-2) \int_0^\infty |v_t|^2 dt. \end{aligned}$$

Applying the Bliss lemma for $m=2$, $n=2_k(s) = \frac{2(d_k-s)}{d_k-2}$, $r = \frac{2-s}{d_k-2}$

$$J = \int_0^\infty |v_t|^2 dt$$

and

$$I = \int_0^\infty |v(t)|^{2_k(s)} t^{\frac{2d_k-s-2}{2-d_k}} dt = \int_0^\infty |v(t)|^{2_k(s)} t^{\frac{2-s}{d_k-2} - \frac{d_k-s}{d_k-2} 2} dt,$$

we get

$$I \leq \frac{d_k-2}{d_k-s} \left[\frac{\frac{2-s}{d_k-2} \Gamma\left(\frac{2(d_k-s)}{2-s}\right)}{\Gamma\left(\frac{d_k-2}{2-s}\right) \Gamma\left(\frac{2(d_k-s)-d_k-2}{2-s}\right)} \right]^{\frac{2-s}{d_k-2}} J^{\frac{d_k-s}{d_k-2}}.$$

Therefore

$$\begin{aligned}
 A &= \frac{1}{d_k - 2} I \\
 &\leq \frac{1}{d_k - 2} \frac{d_k - 2}{d_k - s} \left[\frac{\frac{2-s}{d_k-2} \Gamma\left(\frac{2(d_k-s)}{2-s}\right)}{\Gamma\left(\frac{d_k-2}{2-s}\right) \Gamma\left(\frac{2(d_k-s)-(d_k-2)}{2-s}\right)} \right]^{\frac{p-s}{d_k-p}} \left(B \frac{1}{d_k - 2} \right)^{\frac{d_k-s}{d_k-2}} \\
 &= \frac{1}{(d_k - s)} \left[\frac{\frac{2-s}{d_k-2} \Gamma\left(\frac{2(d_k-s)}{2-s}\right)}{\Gamma\left(\frac{d_k-2}{2-s}\right) \Gamma\left(\frac{2(d_k-s)-(d_k-2)}{2-s}\right)} \right]^{\frac{p-s}{d_k-p}} \left(\frac{1}{d_k - 2} \right)^{\frac{d_k-s}{d_k-2}} B^{\frac{d_k-s}{d_k-2}}.
 \end{aligned}$$

Hence, we get

$$\begin{aligned}
 HS_{2,s,k} &= \inf \frac{\int_{\mathbb{R}_\epsilon^N} |\nabla u|^2 h_k(x) dx}{\left(\int_{\mathbb{R}_\epsilon^N} \frac{|u(x)|^{2k(s)}}{|x|^s} h_k(x) dx \right)^{\frac{d_k-2}{d_k-s}}} \\
 &= \inf \frac{\left(\int_{\mathbb{S}_\epsilon^{N-1}} h_k(\sigma) d\sigma \right) \int_0^\infty |u_r|^2 r^{d_k-1} dr}{\left(\int_{\mathbb{S}_\epsilon^{N-1}} h_k(\sigma) d\sigma \right)^{\frac{d_k-2}{d_k-s}} \left(\int_0^\infty |u(r)|^{2k(s)} r^{d_k-1-s} dr \right)^{\frac{d_k-2}{d_k-s}}} \\
 &= \inf \left(\int_{\mathbb{S}_\epsilon^{N-1}} h_k(\sigma) d\sigma \right)^{\frac{2-s}{d_k-s}} \frac{B}{A^{\frac{d_k-2}{d_k-s}}} \\
 &= \inf \left(\int_{\mathbb{S}_\epsilon^{N-1}} h_k(\sigma) d\sigma \right)^{\frac{2-s}{d_k-s}} \left(\frac{B^{\frac{d_k-s}{d_k-2}}}{A} \right)^{\frac{d_k-2}{d_k-s}} \\
 &= \left(\int_{\mathbb{S}_\epsilon^{N-1}} h_k(\sigma) d\sigma \right)^{\frac{2-s}{d_k-s}} \left[(d_k - s) \left[\frac{\Gamma\left(\frac{d_k-2}{2-s}\right) \Gamma\left(\frac{(d_k-s)p-d_k-2}{2-s}\right)}{\frac{2-s}{d_k-2} \Gamma\left(\frac{2(d_k-s)}{2-s}\right)} \right]^{\frac{2-s}{d_k-2}} (d_k - 2)^{\frac{d_k-s}{d_k-2}} \right]^{\frac{d_k-2}{d_k-s}}.
 \end{aligned}$$

The infimum can be achieved at functions of the form $v(t) = \int_0^t a \left(b \lambda^{\frac{2-s}{d_k-2}} + 1 \right)^{-\frac{d_k-s}{2-s}} d\lambda = a \left(b + t^{-\frac{2-s}{d_k-2}} \right)^{-\frac{d_k-2}{2-s}}$, $a \in \mathbb{R}$, $b > 0$. That is

$$u(r) = v(r^{2-d_k}) = a \left(b + r^{-(2-d_k)\frac{2-s}{d_k-2}} \right)^{-\frac{d_k-2}{2-s}} = a \left(b + r^{2-s} \right)^{-\frac{d_k-2}{2-s}}.$$

We now compute $\int_{\mathbb{S}_\epsilon^{N-1}} h_k(\sigma) d\sigma$. This has been done in [33]. For the completeness, we will provide the computation here.

We first note that $\int_{\mathbb{S}_\epsilon^{N-1}} h_k(\sigma) d\sigma$ does not depend on the Weyl chamber. Also, the number of Weyl chambers equals the order of the reflection group G . Therefore

$$\int_{\mathbb{S}_\epsilon^{N-1}} h_k(\sigma) d\sigma = \frac{1}{|G|} \int_{\mathbb{S}^{N-1}} h_k(\sigma) d\sigma.$$

Now, consider the Macdonald-Mehta integral associated to R

$$M_k = \int_{\mathbb{R}^N} e^{-\frac{|x|^2}{2}} h_k(x) dx.$$

It was showed in [13] that

$$M_k = (2\pi)^{\frac{N}{2}} \prod_{\alpha \in R_+} \frac{\Gamma\left(k(\alpha) + \frac{1}{2} \langle \rho_k, \alpha \rangle + 1\right)}{\Gamma\left(\frac{1}{2} \langle \rho_k, \alpha \rangle + 1\right)},$$

where $\rho_k = \sum_{\alpha \in R_+} k(\alpha) \alpha$.

By using the polar coordinate, we get

$$\begin{aligned} M_k &= \int_{\mathbb{S}^{N-1}} h_k(\sigma) d\sigma \int_0^\infty e^{-\frac{r^2}{2}} r^{d_k-1} dr \\ &= 2^{\frac{d_k}{2}-1} \Gamma\left(\frac{d_k}{2}\right) \int_{\mathbb{S}^{N-1}} h_k(\sigma) d\sigma. \end{aligned}$$

Therefore

$$\int_{\mathbb{S}^{N-1}} h_k(\sigma) d\sigma = \frac{1}{|G|} \frac{(2\pi)^{\frac{N}{2}}}{2^{\frac{d_k}{2}-1} \Gamma\left(\frac{d_k}{2}\right)} \prod_{\alpha \in R_+} \frac{\Gamma\left(k(\alpha) + \frac{1}{2} \langle \rho_k, \alpha \rangle + 1\right)}{\Gamma\left(\frac{1}{2} \langle \rho_k, \alpha \rangle + 1\right)}.$$

□

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